Pre-symplectic structure on the space of connections

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§1. Introduction

Let X be an oriented Riemannian four-manifold with boundary $M = \partial X$.
For the trivial principal bundle $P = X \times SU(n)$ we denote by $\mathcal{A}(X)$ the space of irrreducible connections on X. We shall prove the following theorems.

Theorem

Let $P = X \times SU(n)$ be the trivial $SU(n)$–principal bundle on a four-manifold X. There exists a canonical pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ given by the 2-form

$$\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X Tr[(ab - ba)F_A] - \frac{1}{24\pi^3}\int_M Tr[(ab - ba)A]\,, \quad (1)$$

for $a, b \in T_A\mathcal{A}(X)$

Theorem

Let ω be a 2-form on $\mathcal{A}(M)$ defined by

$$\omega_A(a,b) = -\frac{1}{24\pi^3}\int_M Tr[(ab - ba)A], \tag{2}$$

for $a, b \in T_A\mathcal{A}(M)$.
Let

$$\mathcal{A}_0^\flat(M) = \left\{A \in \mathcal{A}(M);\quad F_A = 0, \quad \int_M Tr\, A^3 = 0\right\} \tag{3}$$

Then $\left(\mathcal{A}_0^\flat(M),\ \omega|_{\mathcal{A}_0^\flat(M)}\right)$ is a pre-symplectic manifold.

This is a part of the author's research on geometric quantization theory of connection spaces.
The followings were proved previously
Kori, T., *Chern-Simons pre-quantization over four-manifolds*, Diff. Geom. and its Appl. 29 (2011), 670-684.

Theorem

Let $\mathcal{G}_0(X)$ be the group of gauge transformations on X that are identity on the boundary M. The action of $\mathcal{G}_0(X)$ on $\mathcal{A}(X)$ is a Hamiltonian action and the corresponding moment map is given by

$$\Phi \quad : \quad \mathcal{A}(X) \longrightarrow (Lie\,\mathcal{G}_0)^* = \Omega^4(X, Lie\,G) : \qquad A \longrightarrow F_A^2 .$$

$$\langle \Phi(A), \xi \rangle \quad = \quad \Phi^{\xi}(A) = \frac{1}{8\pi^3} \int_X Tr(F_A^2 \xi), \ \textit{for}\ \xi \in Lie\,\mathcal{G}_0(X) . \qquad (4)$$

Note

Pre-quantization of a manifold endowed with a closed 2-form [Guillemin et al.] .

For a manifold X endowed with a closed 2-form σ, we call a *pre-quantization* of (X,σ) a hermitian line bundle $(\mathbf{L}, <\,,\,>)$ over X equiped with a hermitian connection ∇ whose curvature is σ.

Theorem

There exists a pre-quantization of the moduli space $\left(\mathcal{M}^\flat = \mathcal{A}^\flat(X)/\mathcal{G}_0(X),\, \omega\right)$, *that is,*
there exists a hermitian line bundle with connection $\mathcal{L}^\flat \longrightarrow \mathcal{M}^\flat$,
whose curvature is equal to the pre-symplectic form $i\,\omega$,

where

$$\mathcal{A}^\flat(X) = \{A \in \mathcal{A}(X); \quad F_A = 0\,,\}$$

and the closed 2-form ω on $\mathcal{M}^\flat$ is induced from that on $\mathcal{A}^\flat(X)$ (as the boundary value and as the quotient) of σ^s:

$$\omega_A(a,b) = -\frac{1}{24\pi^3}\int_M Tr[(ab - ba)A]\,.$$

§2. Space of connections

M : a compact, connected and oriented m-dimensional riemannian manifold
with boundary ∂M. $G = SU(N)$, $N \geq 2$.
$P \xrightarrow{\pi} M$: a principal G-bundle,
$\mathcal{A} = \mathcal{A}(M)$ the space of *irreducible* connections over P,

$$T_A\mathcal{A} = \Omega^1(M, Lie\, G)) : \quad \text{tangent space at } \mathcal{A},$$

and,

$$A \in \mathcal{A}, \quad a \in T_A\mathcal{A} \Longrightarrow A + a \in \mathcal{A}.$$

$$T_A^*\mathcal{A} = \Omega^{m-1}(M, Lie\, G)), \quad \text{cotangent space of at } A$$

The pairing of $\alpha \in T_A^*\mathcal{A}$ and $a \in T_A\mathcal{A}$ is given by

$$\langle \alpha, a \rangle_A = \int_M tr(a \wedge \alpha)$$

A vector field $\mathbf{v}$ on $\mathcal{A}$ is a section of the tangent bundle;
$\mathbf{v}(A) \in T_A\mathcal{A}$,
a 1-form φ on $\mathcal{A}$ is a section of the cotangent bundle; $\varphi(A) \in T_A^*\mathcal{A}$.
For a function $F = F(A)$ on $\mathcal{A}$ valued in a vector space V,
the derivation $\partial_A F$ is defined by the functional variation of $A \in \mathcal{A}$:

$$\partial_A F \quad : \quad T_A\mathcal{A} \longrightarrow V\,, \tag{5}$$

$$(\partial_A F)a \quad = \quad \lim_{t \longrightarrow 0} \frac{1}{t}\left(F(A + ta) - F(A) \right), \qquad \text{for } a \in T_A\mathcal{A}. \tag{6}$$

For example,

$$(\partial_A A)\, a = a,$$

The curvature of $A \in \mathcal{A}$ is by definition

$$F_A = dA + \frac{1}{2}[A \wedge A] \in \Omega^2_{s-2}(M, Lie\, G),$$

and we have

$$(\partial_A F_A)a = d_A a.$$

The derivations of a vector field and a 1-form φ are defined :

$$(\partial_A \mathbf{v})a \in T_A\mathcal{A}, \qquad (\partial_A \varphi)a \in T_A^*\mathcal{A}, \qquad \forall a \in T_A\mathcal{A}.$$

We have the following formulas:

$$[\,\mathbf{v},\ \mathbf{w}\,]_A = (\partial_A \mathbf{v})\mathbf{w}_A - (\partial_A \mathbf{w})\mathbf{v}_A, \tag{7}$$

$$(\mathbf{v}\langle\varphi, \mathbf{u}\rangle)_A = \langle\varphi_A, (\partial_A \mathbf{u})\mathbf{v}_A\rangle + \langle(\partial_A\varphi)\,\mathbf{v}_A, \mathbf{u}_A\rangle. \tag{8}$$

Let $\widetilde{d}$ be the exterior derivative on $\mathcal{A}(M)$. For a function F on $\mathcal{A}(M)$, $(\widetilde{d}F)_A\, a = (\partial_A F)\, a$.
For a 1-form Φ on $\mathcal{A}(M)$,

$$\begin{aligned}(\widetilde{d}\Phi)_A(\mathbf{a}, \mathbf{b}) &= (\partial_A < \Phi, \mathbf{b} >)\mathbf{a} - (\partial_A < \Phi, \mathbf{a} >)\mathbf{b} - < \Phi, [\mathbf{a}, \mathbf{b}] > \\ &= < (\partial_A\Phi)\mathbf{a}, \mathbf{b} > - < (\partial_A\Phi)\mathbf{b}, \mathbf{a} >, \end{aligned} \tag{9}$$

For a 2-form φ is a 2-form

$$(\widetilde{d}\,\varphi)_A(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\partial_A\varphi(\mathbf{b}, \mathbf{c}))\mathbf{a} + (\partial_A\varphi(\mathbf{c}, \mathbf{a}))\mathbf{b} + (\partial_A\varphi(\mathbf{a}, \mathbf{b}))\mathbf{c}\,. \tag{10}$$

§3. Canonical structure on $T^*\mathcal{A}$

$T^*\mathcal{A} \xrightarrow{\pi} \mathcal{A}$: the cotangent bundle.
Tangent space to $T^*\mathcal{A}$ at the point $(A, \lambda) \in T^*\mathcal{A}$ is

$$T_{(A,\lambda)}T^*\mathcal{A} = T_A\mathcal{A} \oplus T^*_\lambda\mathcal{A} = \Omega^1(M,\ Lie\,G) \oplus \Omega^{m-1}(M.\,Lie\,G).$$

The canonical 1-form θ on $T^*\mathcal{A}$ is defined by

$$\theta_{(A,\lambda)}(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \langle\, \lambda, \pi_* \begin{pmatrix} a \\ \alpha \end{pmatrix} \rangle_A = \int_M tr\, a \wedge \lambda. \quad \begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}T^*\mathcal{A}$$

1. For a 1-form ϕ on $\mathcal{A}$,
$$\phi^*\theta = \phi. \tag{11}$$
2. The derivation of the 1-form θ is given by
$$\partial_{(A,\lambda)}\,\theta \begin{pmatrix} a \\ \alpha \end{pmatrix} = \alpha\,, \quad \forall \left(\ a = \langle \alpha, a \rangle_{(A,\lambda}\ \right) \in T_{(A,\lambda)}T^*\mathcal{A}\,. \tag{12}$$

In fact,

$$(\partial_{(A,\lambda)}\theta)(\begin{pmatrix} a \\ \alpha \end{pmatrix}) = \lim_{t\longrightarrow 0}\frac{1}{t}\int_M (\,tr\,a\wedge(\lambda+t\alpha) - tr\,a\wedge\lambda\,) = \int_M tr\,a\wedge\alpha.$$

The canonical 2-form is defind by

$$\sigma = \widetilde{d}\theta. \tag{13}$$

We have

1.

$$\sigma_{(A,\lambda)}\left(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix}\right) = \langle \alpha, b \rangle_A - \langle \beta, a \rangle_A = \int_M tr[\, b\wedge\alpha - a\wedge\beta\,] .$$

In fact

$$\begin{aligned} \widetilde{d}\theta_{(A,\lambda)}\left(\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix}\right) &= \langle \partial_{(A,\lambda)}\theta\begin{pmatrix} a \\ \alpha \end{pmatrix}, \begin{pmatrix} b \\ \beta \end{pmatrix}\rangle - \langle \partial_{(A,\lambda)}\theta\begin{pmatrix} b \\ \beta \end{pmatrix}, \begin{pmatrix} a \\ \alpha \end{pmatrix} \\ &= \langle \alpha, b\rangle_A - \langle \beta, a\rangle_A \end{aligned}$$

2. σ is a *non-degenerate* closed 2-form on the cotangent space $T^*\mathcal{A}$.

For a function $\Phi = \Phi(A, \lambda)$ on $T^*\mathcal{A}$ corresponds the Hamitonian vector field X_Φ

$$(\widetilde{d}\,\Phi\,)_{(A,\lambda)} = \sigma(X_\Phi(A, \lambda), \quad \cdot \quad). \tag{14}$$

The directional derivative $\delta_A\Phi \in T^*\mathcal{A}$ of $\Phi = \Phi(A, \lambda)$ at $(A, \lambda) \in T^*\mathcal{A}$:

$$\langle \delta_A\Phi,\, a\rangle_A = \lim_{t\longrightarrow 0} \frac{1}{t}(\Phi(A + ta, \lambda) - \Phi(A, \lambda)), \quad a \in T_A\mathcal{A}.$$

The exterior differential of Φ at the point (A, λ) is defined.

$$(\widetilde{d}\,\Phi\,)_{(A,\lambda)}\begin{pmatrix} a \\ \alpha \end{pmatrix} = \langle \delta_A\Phi,\, a\rangle_A + \langle \alpha\,,\, \delta_\lambda\Phi\rangle_A \quad \begin{pmatrix} a \\ \alpha \end{pmatrix} \in T_{(A,\lambda)}T^*\mathcal{A}$$

So the Hamiltonian vector field of Φ is

$$X_\Phi = \begin{pmatrix} -\delta_\lambda\Phi \\ \delta_A\Phi \end{pmatrix}.$$

$\mathcal{G}(M)$; Group of (pointed) gauge transformations :

$$\mathcal{G}(M) = \{g \in \Omega^0_s(M, G); \quad g(p_0) = 1\}. \tag{15}$$

$\mathcal{G}(M)$ acts freely on $\mathcal{A}(M)$ by

$$g \cdot A = g^{-1}dg + g^{-1}Ag = A + g^{-1}d_A g. \tag{16}$$

$\mathcal{G}(M) = \Omega^0_s(M, Lie\, G)$ acts on $T_A\mathcal{A}$ by ; $a \longrightarrow Ad_{g^{-1}}\, a = g^{-1}ag$,
on $T^*_A\mathcal{A}$ by its dual $\alpha \longrightarrow g\alpha g^{-1}$.
Hence the canonical 1-form and 2-form are $\mathcal{G}(M)$-invariant.
The infinitesimal action of $\xi \in Lie\, \mathcal{G}(M)$ on $T^*\mathcal{A}$ gives a vector field $\xi_{T^*\mathcal{A}}$ (called fundamental vector field) on $T^*\mathcal{A}$:

$$\xi_{T^*\mathcal{A}}(A, \lambda) = \frac{d}{dt} \exp\, t\xi \cdot \begin{pmatrix} A \\ \lambda \end{pmatrix} = \begin{pmatrix} d_A\xi \\ [\xi, \lambda] \end{pmatrix}, \tag{17}$$

at $(A, \lambda) \in T^*\mathcal{A}$.

The moment map of the action of $\mathcal{G}$ on the symplectic space $(T^*\mathcal{A}, \sigma)$ is described as follows.
For each $\xi \in Lie\,\mathcal{G}$ we define the function

$$\Phi^{\xi}(A,\lambda) = \theta_{(A,\lambda)}(\xi_{T^*\mathcal{A}}) = \int_M tr\,(d_A\xi \wedge \lambda)\,. \qquad (18)$$

Then the correspondence $\xi \longrightarrow \Phi^{\xi}(A,\lambda)$ is linear.
Hence $\Phi(A,\lambda) \in (\,Lie\,\mathcal{G}\,)^*$ and
we have a map

$$\Phi : T^*\mathcal{A} \ni (A,\lambda) \longrightarrow \Phi(A,\lambda) \in (Lie\,\mathcal{G})^*.$$

(18) yields

$$\widetilde{d}\,\Phi^{\xi} = \sigma(\,\xi_{T^*\mathcal{A}}\,,\quad\cdot\quad),\qquad \text{for } \forall\xi \in Lie\,\mathcal{G}. \qquad (19)$$

Theorem

The action of the group of gauge transformations $\mathcal{G}(M)$ on the symplectic space $(\, T^*\mathcal{A}(M),\, \sigma\,)$ is an hamiltonian action and the moment map is given by

$$\Phi^{\xi}(A, \lambda) = \int_M tr\,(\, d_A\xi \wedge \lambda) \,. \tag{20}$$

Generating functions

Let

$$\tilde{s} : \mathcal{A} \longrightarrow T^*\mathcal{A} \qquad : \text{a local section of } T^*\mathcal{A}$$

We write it by $\tilde{s}(A) = (A,\ s(A))$ with $s(A) \in T^*_A\mathcal{A}$.
The pullback of the canonical 1-form θ by $\tilde{s}$ defines a 1-form θ^s on $\mathcal{A}$:

$$\theta^s_A(a) = (\widetilde{s}^*\,\theta)_A a, \qquad a \in T_A\mathcal{A}\,. \tag{21}$$

Lemma

$$\theta^s = s. \tag{22}$$

That is,

$$(\theta^s)_A a = \langle s(A),\ a\rangle\,. \tag{23}$$

for $a \in T_A\mathcal{A}$.

Let $\sigma^s = \tilde{s}^*\sigma$ be the pullback by $\tilde{s}$ of the canonical 2-form σ.

$$\sigma^s_A(a,b) = \sigma_{\tilde{s}(A)}(\tilde{s}_*a,\ \tilde{s}_*b) = \sigma_{(A,s(A))}\left(\begin{pmatrix} a \\ (s_*)_A a \end{pmatrix}, \begin{pmatrix} b \\ (s_*)_A b \end{pmatrix}\right) \tag{24}$$

σ^s is a closed 2-form on $\mathcal{A}$. From Lemma 6 we see

$$\sigma^s = \widetilde{d}\, s\,. \tag{25}$$

Example[(Atiyah-Bott, 1982)]
Let M be a surface (2-dimensional manifold).

$$T_A\mathcal{A} \simeq\ T^*_A\mathcal{A} \simeq \Omega^1(M, LieG)$$

Define the generating function

$$s:\ \mathcal{A} \ni A \longrightarrow s(A) = A \in \Omega^1(M, LieG) = T^*_A\mathcal{A}$$

Then

$$(\theta^s)_A a = \int_M tr(\,Aa\,)\,,$$

and

$$\begin{aligned} \omega_A(a,b) &\equiv \sigma^s_A(a,b) = (\widetilde{d}\,\theta^s)_A(a,b) = \langle(\partial_A\theta^s)a, b\rangle - \langle(\partial_A\theta^s)b, a\rangle \\ &= \int_M tr(ba) - \int_M tr(ab) = 2\int_M tr(ba). \end{aligned} \tag{26}$$

Then $(\mathcal{A}(M), \omega)$ is a symplectic manifold, in fact ω is non-degenerate.

§4. Pre-symplectic structure on the space of connections on a four-manifold

X : Riemannian four-manifold with boundary $M = \partial X$ that may be empty.
$P = X \times SU(n)$: the trivial principal bundle
$\mathcal{A}(X)$: the space of irrreducible $L^2_{s-\frac{1}{2}}$-connections

$$T_A\mathcal{A}(X) = \Omega^1_{s-\frac{1}{2}}(X, Lie\, G)\,, \qquad \text{the tangent space}$$

$\tilde{s}$: a section of the cotangent bundle

$$\tilde{s}(A) = (A, s(A)) = \left(A\,,\; q(\,AF_A + F_AA - \frac{1}{2}A^3)\right)\,. \tag{27}$$

$s(A) = q(AF_A + F_AA - \frac{1}{2}A^3)$: a 3-form on X valued in $su(n)$,
$q_3 = \frac{1}{24\pi^3}$.

The differential of $\tilde{s}$ becomes

$$(\tilde{s}_*)_A\, a = \begin{pmatrix} a \\ q(\,aF_A + F_A\, a + A\, d_A a + d_A a\, A - \frac{1}{2}(aA^2 + AaA + A^2 a)\,) \end{pmatrix},$$

for any $a \in T_A\mathcal{A}$.

Lemma

Let $\theta^s = \tilde{s}^*\theta$ and $\sigma^s = \tilde{s}^*\sigma$ be the pullback of the canonical forms by $\tilde{s}$. Then we have

$$\theta^s_A(a) = \frac{1}{24\pi^3}\int_X Tr[\,(AF + FA - \frac{1}{2}A^3)\,a\,], \quad a \in T_A\mathcal{A}\,, \tag{28}$$

and

$$\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X Tr[(ab - ba)F] - \frac{1}{24\pi^3}\int_{\partial M} Tr[(ab - ba)A]\,. \tag{29}$$

The first equation follows from the deinition; $(\widetilde{s}^*\theta)_A a = \langle s(A), a\rangle$.

For $a, b \in T_A\mathcal{A}$,

$$
\begin{aligned}
(\widetilde{d}\,\theta^s)_A(a,b) &= \langle(\partial_A\theta^s)a, b\rangle - \langle(\partial_A\theta^s)b, a\rangle \\
&= \frac{1}{24\pi^3}\int_X Tr[\,2(ab-ba)F - (ab-ba)A^2 \\
&\qquad - (bd_Aa + d_Aab - d_Aba - ad_Ab)A\,].
\end{aligned}
$$

Since

$$
d\,Tr[(ab-ba)A] = Tr[(b\,d_Aa + d_Aa\,b - d_Ab\,a - a\,d_Ab)A] + Tr[(ab-ba)(F+A^2)],
$$

we have

$$
\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X Tr[\,(ab-ba)F\,] - \frac{1}{24\pi^3}\int_M Tr[\,(ab-ba)A\,], \tag{30}
$$

for $a, b \in T_A\mathcal{A}$.

Theorem

Let $P = X \times SU(n)$ be the trivial $SU(n)$–principal bundle on a four-manifold X. There exists a pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ given by the 2-form

$$\sigma^S_A(a,b) = \frac{1}{8\pi^3}\int_X Tr[(ab-ba)F] - \frac{1}{24\pi^3}\int_M Tr[(ab-ba)A]\,. \quad (31)$$

If X has no boundary and A is a flat connection then $\sigma^s_A = 0$, so we have the following

Proposition

Let X be a compact 4-manifold without boundary then

$$L^s = \{\, \tilde{s}(A)\,; \quad A \in \mathcal{A}^\flat(X)\}$$

is a Lagrangian submanifold of $T^*\mathcal{A}(X)$.

In fact $\partial_A \tilde{s}$ is an isomorphism, so $\tilde{s}\mathcal{A}$ becomes a submanifold of $T^*\mathcal{A}$.

§5. Flat connections on a three-manifold

Put

$$\omega_A(a,b) = -q\int_M Tr[(ab-ba)A], \tag{32}$$

$$\kappa_A(a,b,c) = -3q\int_M Tr[(ab-ba)c], \tag{33}$$

for $a,\ b\in T_A\mathcal{A}$.
Then

$$\widetilde{d}\,\omega_A = \kappa_A. \tag{34}$$

In fact, for $a,b,c\in T_A\mathcal{A}$, we have

$$\widetilde{d}\,\omega_A(a,b,c) = 3\partial_A(\omega_A(a,b))(c) = -3q\int_M Tr[(ab-ba)c] = \kappa_A(a,b,c).$$

1. $\kappa_A = 0 \quad \Longleftrightarrow \quad (\mathcal{A}(M), \omega)$ is pre-symplectic.
2. In general $(\mathcal{A}(M), \omega)$ is not pre-symplectic.
   Which subspace of $(\mathcal{A}(M)$ is pre-symplectic?
   For $SU(2)$, it is shown that

$$\kappa \equiv 0, \, \omega \equiv 0.$$

In the following we deal with the case for $G = SU(n)$, $n \geq 3$.

Is the space of flat connections $\mathcal{A}^\flat(M)$ pre-symplectic ?
Let $\mathcal{A}^\flat = \mathcal{A}^\flat(M)$ be the space of flat connections;

$$\mathcal{A}^\flat(M) = \{A \in \mathcal{A}(M);\ F_A = 0\}.$$

The tangent space of $\mathcal{A}^\flat$ at $A \in \mathcal{A}^\flat$ is given by

$$T_A\mathcal{A}^\flat = \{a \in \Omega^1(M, Lie\, G); \quad d_A a = 0\}, \tag{35}$$

1. orthogonally decomposition:

$$T_A\mathcal{A}^\flat = \{d_A\xi;\ \xi \in \mathcal{G}(M)\} \oplus H_A^\flat,$$

where $\quad H_A^\flat = \{a \in \Omega^1(M, ad\, P);\ d_A^* a = d_A a = 0\}.$

2. $\mathcal{A}^\flat(M)$ is $\mathcal{G}(M)$–invariant,
3. $d_A\xi$ for $\xi \in Lie\,\mathcal{G}(M)$ is a vector field along $\mathcal{A}^\flat(M)$,
4. $d_A\, d_A\xi = [F_A, \xi] = 0$,
i.e. the action of $\mathcal{G}(M)$ on $\mathcal{A}^\flat(M)$ is infinitesimally symplectic.

Direct computation to show that $(\mathcal{A}^\flat(M), \omega)$ *, or its subspace, is pre-symplectic is difficult.*
For example, if we take the following section of the cotangent bundle $T^*_A\mathcal{A} \simeq \Omega^2(M,\, Lie\, G)$;

$$\tilde{f}(A) = (A,\, F_A\,),$$

then

$$\begin{aligned}\sigma^f(a,b) &= \sigma_{(A,F_A)}\left(\begin{pmatrix} a \\ d_A a \end{pmatrix}, \begin{pmatrix} b \\ d_A b \end{pmatrix}\right) \\ &= \int_M tr\,(b \wedge d_A a - a \wedge d_A b) = \int_M d\,(tr\,(ab)) = 0.\end{aligned}$$

so $\widetilde{d}\,F = 0$. Every connection is a critical point of the generating function F.

Next if we take

$$\tilde{t}(A) = (A,\ A^2\ ).$$

we have

$$\sigma^t(a,b) = \sigma_{(A,A^2)}(\begin{pmatrix} a \\ aA + Aa \end{pmatrix}, \begin{pmatrix} b \\ bA + Ab \end{pmatrix}) \ = 0.$$

Thus the pullback of the canonical 2-form σ by the local section $s(A) = pF_A + qA^2$ gives no effective 2-form on $\mathcal{A}(M)$. Nevertheless Theorem **??** presents a 2-form on $\mathcal{A}(M)$ that is related to the boundary restriction of the canonical pre-symplectic form σ^s on $\mathcal{A}(X)$ for a four-manifold X that cobord M.
Things being so we compare them with the pre-symplectic space $(\mathcal{A}^\flat(X), \sigma^s|\mathcal{A}^\flat(X))$ *over a 4-manifold* X *that cobords* M.

Theorem

A pre-symplectic structure on the space of irreducible connections $\mathcal{A}(X)$ is given by the 2-form

$$\sigma^s_A(a,b) = \frac{1}{8\pi^3}\int_X Tr[(ab-ba)F_A] - \frac{1}{24\pi^3}\int_M Tr[(ab-ba)A]\,,\ a,b \in T_A\mathcal{A}(X) \tag{36}$$

Corollary

Let $\mathcal{A}^\flat(X) = \{A \in \mathcal{A}(X);\quad F_A = 0,\quad \}$.
Then $\left(\mathcal{A}^\flat(X),\ \omega|_{\mathcal{A}^\flat(X)}\right)$ *is a pre-symplectic manifold with*

$$\omega_A(a,b) = -\frac{1}{24\pi^3}\int_M Tr[(ab-ba)A]\,,\quad a,b \in T_A\mathcal{A}^\flat(X) \tag{37}$$

Corollary

Let $\mathcal{G}_0(X)$ be the group of gauge transformations on X that are identity on the boundary M.
Let $\mathcal{M}^\flat(X) = \mathcal{A}^\flat(X)/\mathcal{G}_0(X)$, be the moduli space of flat connections on X. Then the closed 2-form $\omega|_{\mathcal{A}^\flat(X)}$ descends to a closed 2-form on $\mathcal{M}^\flat(X)$, hence $\left(\mathcal{M}^\flat(X), \omega\right)$ is a pre-symplectic manifold .

We study the space of connections on a 3-manifold M by looking at the space of connections on a 4-manifold X that cobord M; $\partial X = M$.

M: a 3-manifold ,

P : a principal G–bundle over M

A : a connectionover P,

• **Extension of** M, P, A

$\exists\ X$: an oriented 4-manifold with boundary $\partial X = M$,

$\exists\ \mathbf{P} \longrightarrow X$: a G–bundle with connection $\exists\mathbf{A}$

such that $(\mathbf{P}|M,\ \mathbf{A}|M) = (P, A)$.

$$r_X : \mathcal{A}(X) \longrightarrow \mathcal{A}(M) ; \qquad \text{restriction map to the boundary} \tag{38}$$

$$r_X(A) = A|M, \quad A \in \mathcal{A}(X).$$

The tangent map of r_X at $\mathbf{A} \in \mathcal{A}(X)$ is

$$\rho_{X,\mathbf{A}} : T_{\mathcal{A}}\mathcal{A}(X) = \Omega^1_{s-\frac{1}{2}}(X, Lie\,G) \longrightarrow T_A\mathcal{A}(M) = \Omega^1_{s-1}(M, Lie\,G),$$

$\mathcal{G}(X)$: group of $L^2_{s+\frac{1}{2}}$-gauge transformations on X
$\mathcal{G}(M)$: group of L^2_s-gauge transformations on M .
$\mathcal{G} = \mathcal{G}(M)$ is divided into denumerable sectors labeled by the mapping degree

$$\deg f = \frac{1}{24\pi^2}\int_M Tr\,(df\,f^{-1})^3 . \tag{39}$$

$$\deg(g\,f) = \deg(f) + \deg(g). \tag{40}$$

$$\mathcal{G}_0(X) = \{g \in \mathcal{G}(X);\ g|M = \mathrm{Id}_{\mathcal{G}(M)}\} = ker\{r_X : \mathcal{G}(X) \longrightarrow \mathcal{G}(M)\}$$

If X is simply connected then, for a $f \in \mathcal{G}(M)$,

$$\exists \mathbf{f} \in \mathcal{G}(X) \text{ such that } f = \mathbf{f}|M \Longleftrightarrow deg f = 0 \,.$$

Thus we have the following exact sequence:

$$1 \longrightarrow \mathcal{G}_0(X) \longrightarrow \mathcal{G}(X) \xrightarrow{r_X} \Omega_0^M G \longrightarrow 1, \tag{41}$$

here

$$\Omega_0^M G = \{ g \in \mathcal{G}(M); \quad \deg g = 0 \}.$$

On a 3-manifold any principal bundle has a trivialization.
Fix a trivialization
so that a $su(n)$-connection is identified with a $su(n)$-valued 1-form.
We define the 3-dimensional Chern-Simons function:

$$\mathrm{CS}_{(3)}(A) = \frac{1}{8\pi^2}\int_M Tr\,(AF - \frac{1}{3}A^3), \quad A \in \mathcal{A}(M).$$

For any extension $\mathbf{A} \in \mathcal{A}(X)$ of $A \in \mathcal{A}(M)$; $\mathbf{A}|M = A$, we have

$$\int_X Tr[\,F_{\mathbf{A}}^2] = \int_M Tr[AF_A - \frac{1}{3}A^3]\,.$$

Proposition

For $A \in \mathcal{A}(M)$ and $g \in \mathcal{G}(M)$, we have

$$CS_{(3)}(g\cdot A) = CS_{(3)}(A) + \deg g. \tag{42}$$

X : 4-manifold that cobord M; $\partial X = M$.
$\mathcal{A}(X)$: the space of connections over the trivial bundle $X \times G$.
$\mathcal{A}^\flat(X) = \{A \in \mathcal{A}(X); \quad F_A = 0\}$: the space of flat connections on X.
$\mathcal{A}^\flat(M) = \{A \in \mathcal{A}(M); \quad F_A = 0\}$: the space of flat connections on M.

$$r_X : \mathcal{A}^\flat(X) \longrightarrow \mathcal{A}^\flat(M), \qquad \text{restriction to the boundary}$$

$$r_X(\mathbf{A}) = \mathbf{A}|_M .$$

We show that

- $r_X : \mathcal{A}^\flat(X) \longrightarrow \mathcal{A}^\flat_0(M) = \{A \in \mathcal{A}^\flat(M); \quad CS_{(3)}(A) = 0\}$ is a surjective submersion.

Then we can show easily that κ vanishes on $\mathcal{A}^\flat_0(M)$, that is, $(\mathcal{A}^\flat_0(M), \omega)$ is pre-symplectic (needs a long discussion).

•We shall look at the range of $r_X : \mathcal{A}^\flat(X) \longrightarrow \mathcal{A}^\flat(M)$,
(independent of the cobording 4-manifold X.)

Lemma

$A \in \mathcal{A}^\flat(M)$. If

$$\int_M Tr\, A^3 = 0,$$

there is a $\mathbf{A} \in \mathcal{A}^\flat(X)$ that extends A ; $r_X(\mathbf{A}) = A$.

Proof Let $\widetilde{X}$ be the universal covering of X and $\widetilde{M}$ be the subset of $\widetilde{X}$ that lies over M. Let f_A be the parallel transformation by A along the paths starting from $m_0 \in M$. It defines a smooth map on the covering space $\widetilde{M}$; $f = f_A \in Map(\widetilde{M}, G)$, such that $f^{-1}df = A$. Then the degree of f is equal to

$$\deg f = \frac{1}{24\pi^2}\int_M Tr\, A^3 = \mathrm{CS}_{(3)}(A). \tag{43}$$

If the integral vanishes then $\deg f = 0$ and there is a $\mathbf{f} \in \mathcal{G}(\widetilde{X})$ that extends f. Therefore $\mathbf{A} = \mathbf{f}^{-1}d\mathbf{f} \in \mathcal{A}^\flat(X)$ gives a flat extension of $\mathbf{A}$ over X such that $r_X(\mathbf{A}) = A$.

For $A \in \mathcal{A}^\flat(M)$ and $a \in T_A\mathcal{A}^\flat(M)$, we have

$$(\widetilde{d}\,\mathrm{CS}_{(3)})_A\, a = \frac{1}{8\pi^2}\int_M Tr(A^2 a) = \frac{1}{8\pi^2}\int_M dTr(Aa) = 0. \tag{44}$$

Hence $\mathrm{CS}_{(3)}$ is constant on every connected component of $\mathcal{A}^\flat(M)$.

Definition

For each $k \in \mathbf{Z}$ we define

$$\mathcal{A}^\flat_k(M) = \left\{ A \in \mathcal{A}^\flat(M); \quad \int_M Tr\, A^3 = k \right\}. \tag{45}$$

We call $\mathcal{A}^\flat_k(M)$ the k-sector of the flat connections.

$\mathcal{A}^\flat_k(M)$ is invariant under the action of
$\Omega^M_0 G = \{g \in \mathcal{G}(M);\ \deg g = 0\}$.

Proposition

For any 4-manifold X with the boundary M we have the following properties:

1. *The image of r_X is precisely $\mathcal{A}_0^\flat(M)$.*
2. $d_A(\,Lie\,\mathcal{G}(M)) \in T_A\mathcal{A}_0^\flat(M)$.
3. *The action of the group of gauge transformations $\mathcal{G}(M)$ on $\mathcal{A}_0^\flat(M)$ is infinitesimally symplectic.*

Proof

It follows from the above discussion that any $A \in \mathcal{A}_0^\flat(M)$ is the boundary restriction of a $\mathbf{A} \in \mathcal{A}^\flat(X)$. Conversely let $A = r_X(\mathbf{A})$ for a $\mathbf{A} \in \mathcal{A}^\flat(X)$. Then

$$\int_M Tr\, A^3 = \int_X Tr\, \mathbf{A}^4 = 0,$$

and $A \in \mathcal{A}_0^\flat(M)$. Thus, for any 4-manifold X that cobord M the image of r_X is precisely $\mathcal{A}_0^\flat(M)$. The propperties 2 and 3 are restatement of the facts

$$d_A\xi \in T_A\mathcal{A}^\flat(M), \quad L_{d_A\xi}\,\omega = 0.$$

□

Lemma

Let X be a 4-manifold with $\partial X = M$ then

- r_X *is a submersion.*

Theorem

$(\mathcal{A}_0^{\flat}(M), \omega)$ *is a pre-symplectic manifold.*

We must show

$$\widetilde{d}\,\omega_A = \kappa_A = 0,$$

for any $A \in \mathcal{A}_0^\flat(M)$. Let X be a 4-manifold with boundary $\partial X = M$ and let $\mathbf{P}$ be a G–bundle over X with a connection $\mathbf{A}$ such that $A = r_X \mathbf{A}$.

Let $a, b, c \in T_A \mathcal{A}^\flat(M)$. $\rho_{X,\mathbf{A}}$ being surjective, there are $\mathbf{a}, \mathbf{b}, \mathbf{c} \in T_{\mathbf{A}} \mathcal{A}^\flat(X)$ that extend a, b, c respectively. Then we have

$$\begin{aligned} \kappa_A(a,b,c) &= -q \int_M Tr[(ab - ba)c] \\ &= -q \int_X Tr[\,(d_{\mathbf{A}}\mathbf{a}\,\mathbf{b} - \mathbf{a}\,d_{\mathbf{A}}\mathbf{b} \; - \; d_{\mathbf{A}}\mathbf{b}\,\mathbf{a} \; + \; \mathbf{b}\,d_{\mathbf{A}}\mathbf{a})\,\mathbf{c} \\ &+ \; (\mathbf{ab} - \mathbf{ba})\,d_{\mathbf{A}}\mathbf{c}\,] = 0 \end{aligned} \tag{46}$$

because of $d_{\mathbf{A}}\mathbf{a} = 0$, etc.. $\square$

Let $\mathcal{M}^\flat(X)$ be as was introduced in 1.3 the moduli space of flat connections over X. Because of Theorem **??** $\mathcal{M}^\flat(X)$ is endowed with the pre-symplectic structure

$$\sigma^s_{[\mathbf{A}]}(\mathbf{a},\,\mathbf{b}) = -q\int_M Tr[(ab-ba)A]\,, \tag{47}$$

for $\mathbf{A}\in\mathcal{A}^\flat(X)$ and $\mathbf{a},\,\mathbf{b}\in T_{\mathbf{A}}\mathcal{A}^\flat(X)$, where $A=r_X(\mathbf{A})$ and $a=\rho_X(\mathbf{a})$, $b=\rho_X(\mathbf{b})$. The right hand side is the pre-symplectic form on $\mathcal{A}^\flat_0(M)$ that coincides with $\omega_A(a,b)$.

We have evidently $r_X(g \cdot \mathbf{A}) = r_X(\mathbf{A})$ for $g \in \mathcal{G}_0$. Hence it induces the map

$$\bar{r}_X : \mathcal{M}^\flat(X) \longrightarrow \mathcal{A}^\flat(M). \tag{48}$$

Proposition

$\bar{r}_X$ gives a diffeomorphism of $\mathcal{M}^\flat(X)$ to $\mathcal{A}_0^\flat(M)$.

Proposition

$$\bar{r}_X : \mathcal{M}^\flat(X) \longrightarrow \mathcal{A}_0^\flat(M)$$

gives an isomorphism of pre-symplectic manifolds;

$$\left(\mathcal{M}^\flat(X),\, \sigma^s\right) \simeq \left(\mathcal{A}_0^\flat(M), \omega\right). \tag{49}$$